Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

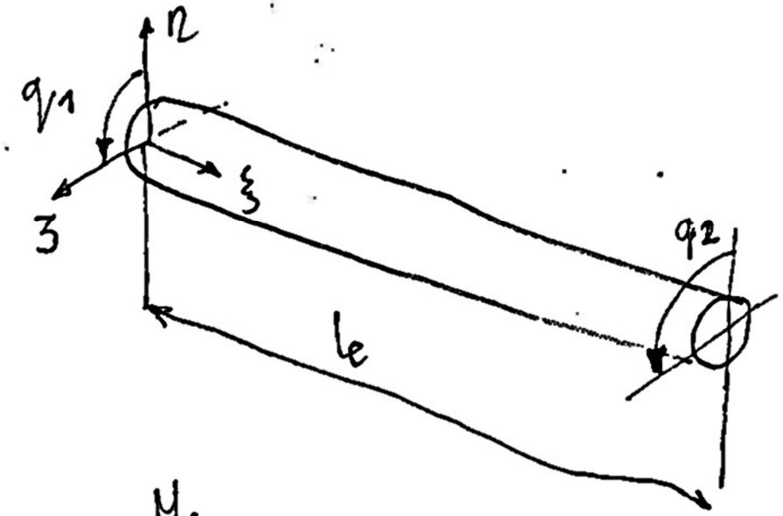
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Torsion bar finite element

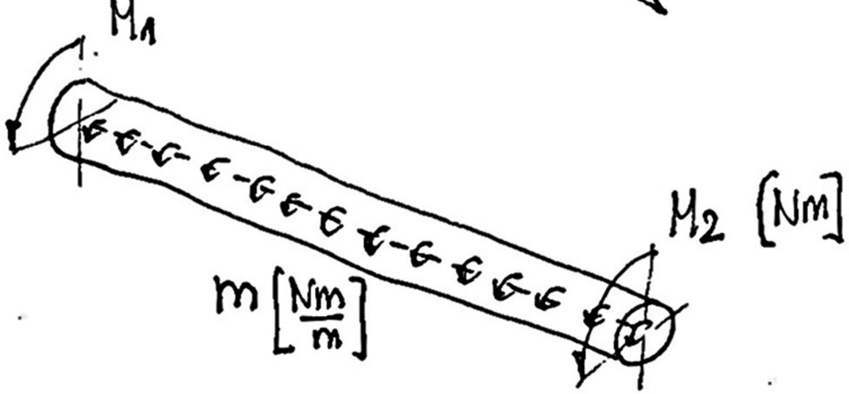
04.2021

TORSION BAR



q_1, q_2 - rotations
(twist angles)

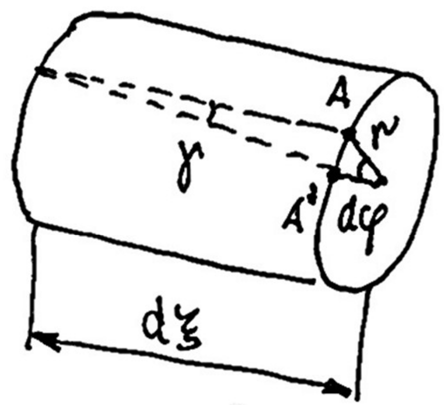
load



M- torque

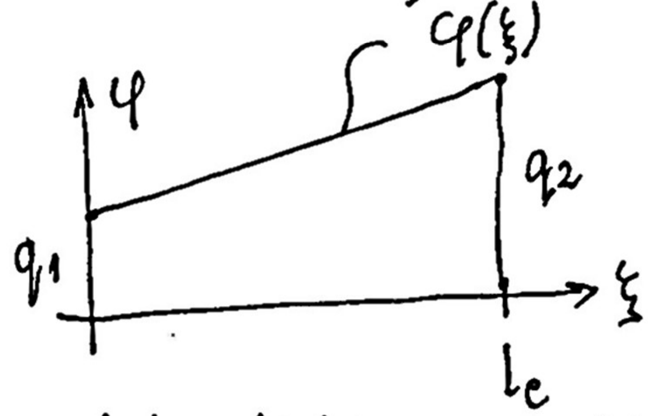
torque per unit lenght

deformation



$$AA' \cong \gamma \cdot d\xi = r\, d\varphi$$

$$\gamma = r \frac{d\varphi}{d\xi}$$



$$N_1(\xi) = 1 - \frac{\xi}{l_e}$$

$$N_2(\xi) = \frac{\xi}{l_e}$$

$$\varphi(\xi) = N_1(\xi)\cdot q_1 + N_2(\xi)\cdot q_2 =$$

$$= \underset{1\times 2}{\lfloor N \rfloor} \cdot \underset{2\times 1}{\{q\}_e}$$

$$\frac{d\varphi}{d\xi} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e} = \left\lfloor -\frac{1}{l_e}, \frac{1}{l_e} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e}$$

elastic strain energy

$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{\Uparrow}{\tau}\cdot\gamma \, d\Omega_e = \frac{1}{2}\int_{\Omega_e} G\cdot\gamma^2 \, d\Omega_e = \frac{1}{2}\int_{\Omega_e} G\, r^2 \left(\frac{d\varphi}{d\xi}\right)^2 d\Omega_e =$$

Hooke's law

$$\tau = G\cdot\gamma = \frac{E}{2(1+\nu)}\cdot\gamma$$

$$= \frac{1}{2}\int_0^{l_e} G\,\frac{d\varphi}{d\xi}\cdot\frac{d\varphi}{d\xi}\cdot\underbrace{\int_A r^2 dA}_{\text{polar moment of inertia } J_s}\, d\xi =$$

$$J_s = \frac{\pi d^4}{32} \text{ for a circle}$$

$$= \frac{1}{2}\int_0^{l_e} GJ_s \cdot \frac{d\varphi}{d\xi} \cdot \frac{d\varphi}{d\xi}\, d\xi =$$

$$\underset{1\times 2}{\lfloor q \rfloor_e} \cdot \begin{Bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \end{Bmatrix} \qquad \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e}$$

$$= \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e \quad ; \quad \text{where:}$$

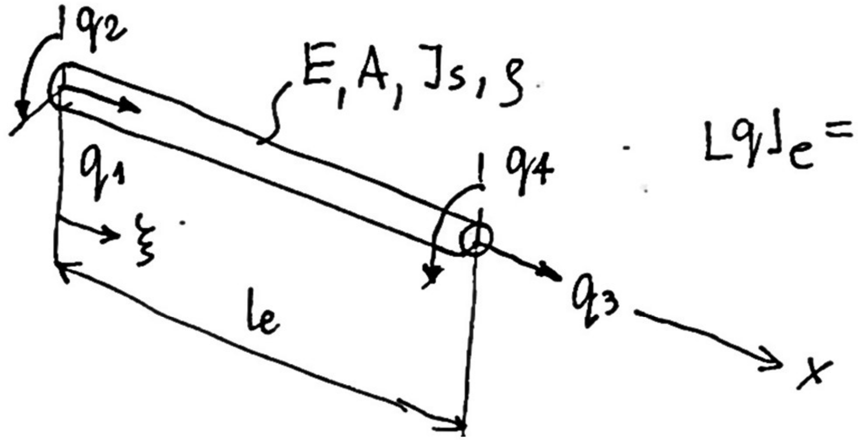
$$\underset{2\times 2}{[k]_e} = GJ_s \begin{bmatrix} \int_0^{l_e} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi}d\xi & \int_0^{l_e} \frac{dN_1}{d\xi}\frac{dN_2}{d\xi}d\xi \\ \int_0^{l_e} \frac{dN_2}{d\xi}\frac{dN_1}{d\xi}d\xi & \int_0^{l_e} \frac{dN_2}{d\xi}\frac{dN_2}{d\xi}d\xi \end{bmatrix} = \frac{GJ_s}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

potential energy of loading

$$W_e = \underset{1\times 2}{\lfloor F \rfloor_e} \cdot \{q\}_e + M_1 \cdot q_1 + M_2 \cdot q_2$$

$$\underset{1\times 2}{\lfloor F \rfloor_e} = \lfloor \int_0^{l_e} m(\xi) \cdot N_1(\xi)\, d\xi \, , \int_0^{l_e} m(\xi) \cdot N_2(\xi)\, d\xi \rfloor$$

AXIAL + TORSION BAR



$$\lfloor q \rfloor_e = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

AXIAL BAR:

$$\underset{2\times 2}{[k]_A} = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lfloor q \rfloor_A = \lfloor q_1, q_3 \rfloor \quad \text{(translational D.O.F)}$$

$$\underset{4\times 4}{[k]_A^*} = \frac{EA}{l_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

TORSION BAR:

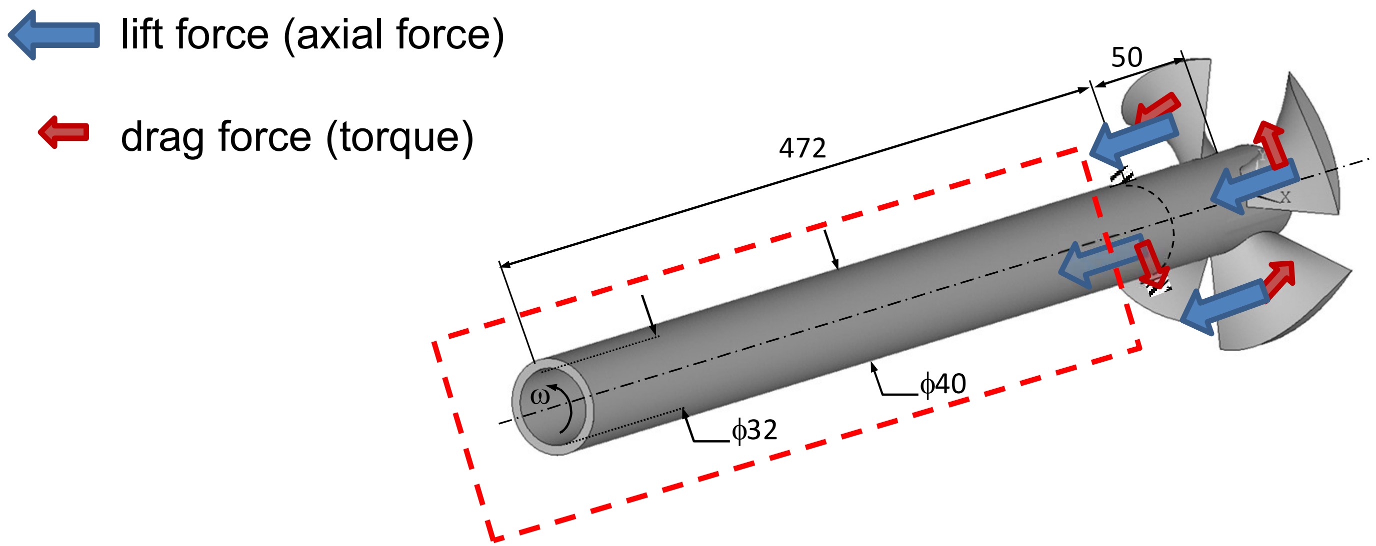
$$\underset{2\times 2}{[k]_T} = \frac{GJ_s}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

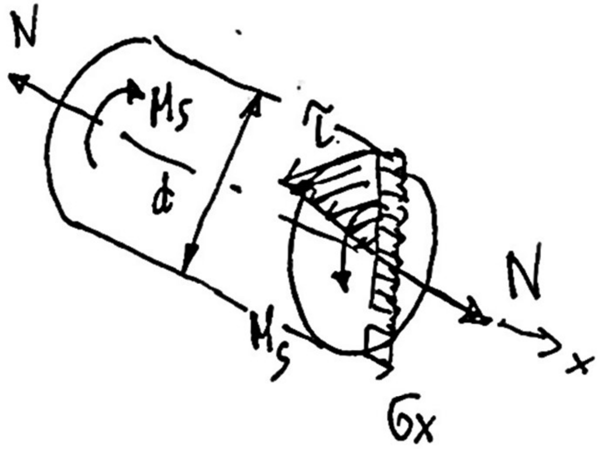
$$\lfloor q \rfloor_T = \lfloor q_2, q_4 \rfloor \quad \text{(rotational D.O.F)}$$

$$\underset{4\times 4}{[k]_T^*} = \frac{GJ_s}{l_e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{4\times 4}{[k]_e} = \underset{4\times 4}{[k]_A^*} + \underset{4\times 4}{[k]_T^*} = \frac{1}{l_e}\begin{bmatrix} EA & 0 & -EA & 0 \\ 0 & GJ_s & 0 & -GJ_s \\ -EA & 0 & EA & 0 \\ 0 & -GJ_s & 0 & GJ_s \end{bmatrix}$$

Example. Axial+torsion element used to model the propeller shaft





$$\sigma_x = E \cdot \varepsilon_x = E \cdot \frac{du}{d\xi} =$$

$$= E \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix}_A =$$

$$= E \, \frac{q_3 - q_1}{l_e}$$

$$\frac{dN_1}{d\xi} = -\frac{1}{l_e} \quad , \quad \frac{dN_2}{d\xi} = \frac{1}{l_e}$$

$$\tau(r) = G \cdot \gamma(r) = G \cdot r \, \frac{d\varphi}{d\xi} = G \cdot r \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_4 \end{Bmatrix}_T =$$

$$= G \cdot \frac{q_4 - q_2}{l_e} \cdot r \qquad , \quad \tau_{max} = \frac{Gd}{2} \cdot \frac{q_4 - q_2}{l_e}$$

$$\sigma_{eqv}^{max} = \sqrt{\sigma_x^2 + 3 \tau_{max}^2}$$